Journal of Statistical Physics, Vol. 25, No. 2, 1981

Field Theory of the Two-Dimensional Ising Model: II. Nonlocal Specific Heat

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Received May 28, 1980

The nonlocal specific heat is calculated for the two-dimensional Ising model and found to be identical to that found by Bray for the four-dimensional model. Furthermore, it is noted that analytic continuation in terms of a spectral function provides an especially simple description of the nonlocal specific heat. From unitarity the spectral function is the rate of pair production in the Minkowski metric, and is calculated to be equal to the velocity of the outgoing particles.

KEY WORDS: Ising model; field theory; specific heat, nonlocality; wave number dependence; spectral function; analytic continuation.

1. INTRODUCTION

The logarithmic divergence of the specific heat of the two-dimensional Ising model has served since its discovery by Onsager[(1)] as a protype for the study of critical phenomena. In an earlier publication we have presented a simple derivation of this divergence using a continuum formulation.[(2)] The latter was derived from the Lieb–Mattis–Schultz[(3)] fermion version of the model. The Ising model specific heat has been very thoroughly studied during the last few decades. Nevertheless, we wish to present in this short note a new derivation of a further aspect of the critical specific heat. This is its nonlocal behavior, as a function of wave number k. The k-dependent specific heat is the Fourier transform of the energy-density–energy-density correlation function, which has been calculated by Hecht.[(4)] The derivation presented here leads to the Fourier transform

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0022-4715/81/0600-0361$03.00/0

directly. As will be seen, certain simple properties of this correlation function show up only in k space and are not evident in configuration space.

The nonlocal specific heat has been measured in recent light-scattering experiments[5,6] for liquid ^{4}He near its λ point, where the divergence of the constant pressure specific heat is close to being logarithmic.[7,8] Unfortunately the three-dimensional theory, needed for comparison with the experimental results, cannot be worked out in closed form. It is therefore useful to calculate the specific heat scaling function in the neighboring limiting cases of two- and four-dimensional space. Bray[9] has done the four-dimensional calculation, while here we present the answer for two dimensions. We find the remarkable result that the two cases are described by one and the same scaling function. In a separate publication[10] the error in applying the Bray function to three dimensions is estimated to amount to only a few percent. To this accuracy the Bray function is found to give a good account of the experimental data.[2]

Our calculation makes use of analytic continuation[12–15] of the specific heat in the complex k^2 plane.[16] By means of unitarity we determine the specific heat in Section II in terms of a spectral function. This is defined along the cut which extends along the k^2 axis beginning at the two-particle threshold. In Section 3 we use Cauchy's theorem to find the scaling function in the physical domain (positive k^2 axis). Section 4 is a brief summary.

2. SPECTRAL FUNCTION

It has been shown[2] that the calculation of the critical temperature dependence of the equilibrium free energy of the two-dimensional Ising model is mathematically equivalent to finding the ground state energy of the one-dimensional Dirac field, described by the Hamiltonian density

$$H(x) = \psi^\dagger(x)\alpha p\psi(x) + \kappa\psi^\dagger(x)\beta\psi(x) \tag{2.1}$$

$\psi(x)$ is a two-component second-quantized spinor operator. α and β are anticommuting Dirac matrices. Because the momentum operator $p = -i\partial/\partial x$ has only one component, α and β are of rank two and can be chosen to be the Pauli spin matrices σ_1 and σ_3, respectively. κ is a mass which is linearly proportional to the temperature and which vanishes at the critical temperature. Because of this linearity the entropy, obtained from

[2]See Ref. 11; the discrepancy reported in this paper disappears when different thermodynamic data, which we believe to be more correct are used.

the free energy by a temperature differentiation, is essentially

$$S = \partial/\partial\kappa \int dx\, H(x) = \int dx\, \psi^\dagger(x)\sigma_3\psi(x) \tag{2.2}$$

The correlation function corresponding to the specific heat is therefore expressed as the time-ordered Green's function

$$G_2(it) \propto \langle TS(t)S(0)\rangle \tag{2.3}$$

The equilibrium correlation as a function of space is found by analytically continuing Eq. (2.3) to negative imaginary values of t.

The spectral function is obtained from Eq. (2.3) for real values of t by carrying out the Fourier transform and extracting its imaginary part. By unitarity this is equal to the rate of production of a pair of particles, as calculated from the "Golden Rule" of time-dependent perturbation theory. The required matrix element can be obtained either from a trace calculation or from explicit expressions for the Dirac spinors. Taking the latter route, we find the positive and negative energy spinors from the single-particle Dirac equation

$$(\sigma_1 p + \kappa\sigma_3)\psi_\pm = \pm\epsilon\psi_\pm \tag{2.4}$$

where the energy eigenvalue has the relativistic form

$$\epsilon = \left(p^2 + \kappa^2\right)^{1/2} \tag{2.5}$$

The solution of Eq. (2.4) is

$$\psi_+ = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.6a}$$

and orthogonality requires

$$\psi_- = \begin{pmatrix} b \\ -a \end{pmatrix} \tag{2.6b}$$

where

$$a = \frac{1}{\sqrt{2}}(1 + \kappa/\epsilon)^{1/2} \tag{2.7a}$$

and

$$b = \frac{1}{\sqrt{2}}(1 - \kappa/\epsilon)^{1/2} \tag{2.7b}$$

reminiscent of the theory of superconductivity. We remark in passing that the spinor formalism employed here provides an efficient framework for the calculation, but it is by no means essential. A straightforward treatment of the pairing effects using a Bogoliubov-type transformation to quasiparticle operators leads to the same results.[3]

[3] For a more conventional approach to the properties of the Ising model see Ref. 16.

The pair-creation matrix element between the initial ground state (with all of the negative-energy single-particle levels filled) and the final particle–antiparticle pair state is

$$S_{fi} = \psi_+^\dagger \sigma_3 \psi_- = 2ab = p/\epsilon = v \tag{2.8}$$

the velocity of the outgoing particles. Here we have substituted from Eqs. (2.6a, b) and (2.7a, b). The density of states per unit energy is given by

$$\rho_f \propto dp/d\epsilon = 1/v \tag{2.9}$$

The desired rate is therefore

$$2\pi\rho_f |S_{fi}|^2 \propto (1/v)v^2 = v \tag{2.10}$$

Identifying this rate with the spectral function we have

$$F = v \tag{2.11}$$

Bray's formula[9] for the nonlocal specific heat can be derived as a production rate for a pair of bosons in (3 + 1)-dimensional space. Compared to Eq. (2.8) the matrix element lacks the factor of p. But the phase space factor of p^2 which has to be added to Eq. (2.9) for the two additional spatial dimensions compensates fully. The result is that Eq. (2.11) also describes the four-dimensional model.

The monotonicity of F is evident from Eq. (2.11) as a function of $u\kappa^2 = -k^2 = (2\epsilon)^2$, the square of the pair energy. Rising from zero at threshold, where the outgoing particles are at rest, the velocity approaches asymptotically the limiting "velocity of light," normalized here to unity. From Eq. (2.5) we can eliminate the velocity in terms of u to obtain, for $u \geqslant 4$,

$$F(u) = \left(1 - \frac{4}{u}\right)^{1/2} \tag{2.12}$$

$[F(u) = 0$ for $u < 4]$.

3. SCALING FUNCTION

The scaling function in its subtracted form is given by Cauchy's theorem as

$$L(x) = -x^2 \int_4^\infty \frac{du}{u(u + x^2)} F(u) \tag{3.1}$$

Upon substitution of Eq. (2.12) the integration is facilitated by the variable change $u = 4\cosh^2\phi$, which yields Bray's formula

$$L(x) = 2 - 2\left(1 + \frac{4}{x^2}\right)^{1/2} \ln\left[\frac{x}{2} + \left(1 + \frac{x^2}{4}\right)^{1/2}\right] \tag{3.2}$$

where the scaling variable is $x = k/\kappa$. Alternatively, we confirm by inspection that Eq. (3.2) has a branch point at $x^2 = -4$ and that the discontinuity across the cut extending from the branch point is

$$\Delta L = -2\pi i\left(1 - \frac{4}{|x^2|}\right)^{1/2} = -2\pi i F(|x^2|) \qquad (3.3)$$

Because of the uniqueness implied by Cauchy's theorem this identification is equivalent to carrying out the integration in Eq. (3.1). One can verify that Eq. (3.2) agrees with the Fourier transform of Hecht's[4] expression for the energy-density–energy-density correlation function.

The properties of Bray's formula are already familiar from his[9] and other work.[11,17,18] It suffices to note that $L(x)$ is a monotonic negative definite function of x. It vanishes at $x = 0$ and for $0 < x \ll 1$ varies as $-x^2/6$. For $x \gg 1$, $L \sim 2 - 2\ln x = -2\ln(k/e\kappa)$, which defines an effective threshold factor $l = e = 2.718\ldots$. The "rule of thumb" for convolution integrals[19] suggests that l is roughly equal to 2, while the two-particle threshold requires $l > 2$.

4. SUMMARY

The nonlocal specific heat for the two-dimensional Ising model has been found to have a very simple spectral function. This is the velocity of the outgoing particles in the Minkowski metric version of the theory. We have noted that the very same spectral function applies to the theory in four dimensions, establishing the remarkable fact that the two- and four-dimensional models have identical nonlocal specific heat. For physical values of the wave number the latter is given by Bray's formula Eq. (3.2).

ACKNOWLEDGMENTS

In closing it is a pleasure to acknowledge helpful discussions with Dr. Jayanta Bhattacharjee and Dr. John Richardson as well as support from the National Science Foundation under Grant No. DMR80-10513 to the University of Maryland and Grant No. PHY77-27084 to the Institute for Theoretical Physics, University of California, Santa Barbara.

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